

IMPROVED L^p -POINCARÉ INEQUALITIES ON THE HYPERBOLIC SPACE

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ABSTRACT. We investigate the possibility of improving the p -Poincaré inequality $\|\nabla_{\mathbb{H}^N} u\|_p \geq \Lambda_p \|u\|_p$ on the hyperbolic space, where $p > 2$ and $\Lambda_p := [(N-1)/p]^p$ is the best constant for which such inequality holds. We prove several different, and independent, improved inequalities, one of which is a Poincaré-Hardy inequality, namely an improvement of the best p -Poincaré inequality in terms of the Hardy weight r^{-p} , r being geodesic distance from a given pole. Certain Hardy-Maz'ya-type inequalities in the Euclidean half-space are also obtained.

1. INTRODUCTION

Let \mathbb{H}^N denote the hyperbolic space of dimension $N \geq 2$, $\nabla_{\mathbb{H}^N}$, $\Delta_{\mathbb{H}^N}$ and $dv_{\mathbb{H}^N}$ its Riemannian gradient, Laplacian and measure, respectively. It is well known that the L^2 spectrum of $-\Delta_{\mathbb{H}^N}$ is bounded away from zero. More precisely one has $\sigma(-\Delta_{\mathbb{H}^N}) = [(N-1)^2/4, +\infty)$. As a byproduct, the quadratic form inequality

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 dv_{\mathbb{H}^N} \geq \frac{(N-1)^2}{4} \int_{\mathbb{H}^N} u^2 dv_{\mathbb{H}^N}$$

holds for all $u \in C_c^\infty(\mathbb{H}^N)$. See e.g. [14] for an elementary proof. Besides, another inequality which one is very familiar within the Euclidean setting, namely *Hardy's inequality*, holds true as well on \mathbb{H}^N , so that one has, at least for $N \geq 3$,

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 dv_{\mathbb{H}^N} \geq \frac{(N-2)^2}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^2} dv_{\mathbb{H}^N},$$

where $r := \varrho(x, x_0)$ denotes geodesic distance from a fixed pole x_0 . In fact, such inequality holds on any Cartan-Hadamard manifold, where the latter are defined as those manifolds which are complete, simply connected and have nonpositive sectional curvatures. See [12] for details. Hardy-type inequalities have been the object of a large amount of research in the past decades, see for example, with no claim of completeness, [3, 4, 8, 9, 10, 11, 13, 15, 16, 18, 21, 22, 23, 25, 27, 30, 32].

A combination of these inequalities was given in [1] and then rediscovered by other methods in [6]. A simplified version of it reads

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 dv_{\mathbb{H}^N} - \frac{(N-1)^2}{4} \int_{\mathbb{H}^N} u^2 dv_{\mathbb{H}^N} \geq \frac{1}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^2} dv_{\mathbb{H}^N} \quad (1.1)$$

for all $u \in C_c^\infty(\mathbb{H}^N)$, and the constants in (1.1) are sharp (the sharpness of the constant $(N-1)^2/4$ in the l.h.s. being obvious), see [6]. The sharpness of related inequalities in more general manifolds and similar improved inequalities of Rellich type, which are again sharp in suitable senses, are also proved in [6]. See also [5] for related higher order Poincaré-Hardy inequalities.

Key words and phrases. p -Poincaré inequality, hyperbolic space, Poincaré-Hardy inequality .

No L^p analogue of (1.1) is known for $p \neq 2$. It is our purpose here to initiate a study of *improved p -Poincaré inequalities* on \mathbb{H}^N , where we take the attitude of looking for improvements of the L^p -gap inequality

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p dv_{\mathbb{H}^N} \geq \left(\frac{N-1}{p} \right)^p \int_{\mathbb{H}^N} |u|^p dv_{\mathbb{H}^N}, \quad (1.2)$$

valid for all $u \in C_c^\infty(\mathbb{H}^N)$, where it is known that the constant $\left(\frac{N-1}{p} \right)^p$ is the best one for such an inequality to hold, see [28] (a simpler proof of this fact will anyway be given below in Lemma 2.2).

In fact, let $-\Delta_{p,\mathbb{H}^N}$ denote the p -Laplacian operator on \mathbb{H}^N , namely

$$\Delta_{p,\mathbb{H}^N} u := \operatorname{div}_{\mathbb{H}^N} (|\nabla_{\mathbb{H}^N} u|^{p-2} \nabla_{\mathbb{H}^N} u) \quad (1.3)$$

It is well-known that \mathbb{H}^N is a p -hyperbolic manifold, i.e., $-\Delta_{p,\mathbb{H}^N}$ admits a positive Green's function by which the validity of a Hardy-type inequality follows. Less evident is the answer to the following question:

Problem. Does there exist a nonnegative, not identically zero weight W such that the following improved Poincaré inequality

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p dv_{\mathbb{H}^N} - \left(\frac{N-1}{p} \right)^p \int_{\mathbb{H}^N} |u|^p dv_{\mathbb{H}^N} \geq \int_{\mathbb{H}^N} W |u|^p dv_{\mathbb{H}^N} \quad (1.4)$$

holds for all $u \in C_c^\infty(\mathbb{H}^N)$?

A first affirmative answer to the above question was given in [7], see formula (5.25) there. In fact, the authors prove the following result:

Proposition 1.1 ([7]). *Let $p \geq 2$ and $N \geq 2$. Set $r := \varrho(x, x_0)$ with $x_0 \in \mathbb{H}^N$ fixed. There exists a radial weight $0 < W = W(r)$ such that for all $u \in C_c^\infty(\mathbb{H}^N)$ there holds*

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p dv_{\mathbb{H}^N} - \left(\frac{N-1}{p} \right)^p \int_{\mathbb{H}^N} |u|^p dv_{\mathbb{H}^N} \geq \int_{\mathbb{H}^N} W |u|^p dv_{\mathbb{H}^N}.$$

Furthermore,

- near x_0 there holds

$$W(r) \underset{r \rightarrow 0}{\sim} \begin{cases} \left(\frac{N-p}{p} \right)^p \frac{1}{r^p} & \text{if } N > p, \\ \left(\frac{n-1}{n} \right)^n \frac{1}{r^n (\log \frac{1}{r})^n} & \text{if } N = p, \\ C \frac{1}{r^{\frac{p(N-1)}{p-1}}} & \text{if } N < p, \end{cases} \quad (1.5)$$

where $C = C(p, N) := \left(\frac{p-1}{p} \right)^p \left(\int_0^\infty (\sinh s)^{-\frac{N-1}{p-1}} ds \right)^{-p}$ for $N < p$.

- Near infinity, setting $\Lambda_p := \left(\frac{N-1}{p} \right)^p$, there holds

$$W(r) = \Lambda_p + A_N e^{-2r} + o(e^{-2r}) \quad \text{as } r \rightarrow \infty.$$

Hence, the given improvement of the Poincaré inequality is stated in terms of a weight which is power-like near a given pole but exponentially decaying at infinity.

In the present paper we construct different examples of weights W for which inequality (1.4) holds and that are slowly decaying at infinity. In any case, due to their asymptotic behavior the weights provided are not globally comparable. For instance, we prove the existence of a weight which is bounded but does not globally vanish at infinity. Finally, in a suitable range of p we improve the Poincaré inequality via the Hardy weight $W = \frac{C}{\varrho^p(x, x_0)}$, where $\varrho(x, x_0)$ is the geodesic distance from $x_0 \in \mathbb{H}^N$ fixed and $C = C(N, p)$ is a positive constant. This choice seems to be the best compromise to capture the non euclidean behavior of inequality (1.4) at infinity without losing too much information at the origin. The techniques applied in the proofs are: hyperbolic symmetrization and p -convex inequalities together with a suitable transformation which uncovers the Poincaré term. Furthermore, super-solution technique and potential inequalities have been exploited.

The paper is organized as follows. In Section 2 we state our main results on \mathbb{H}^N , Theorems 2.2-2.4. Section 3 discusses a related result in the Euclidean half-space, which is the key one to prove some of the results valid on \mathbb{H}^N but can have some independent interest, see Theorem 3.2. Section 4 contains, for the convenience of the reader, a concise proof of Proposition 1.1. Section 5 discusses the proofs of Theorem 3.2 and, consequently, of Theorem 2.2, which is an improvement of the Poincaré inequality in terms of a weight having different asymptotics in different “directions” and, in particular, not vanishing everywhere at infinity. Theorem 2.3, which states a *Hardy-type improvement* of the Poincaré inequality in the spirit of [1], [6], is proven in Section 6. Whereas our final result, namely Theorem 2.4, which deals with an improved inequality on geodesic balls, is proven in Section 7 as a consequence of the more general Theorem 7.1.

2. PRELIMINARIES AND RESULTS

We have mentioned before that inequality (1.2) holds, and that the constant

$$\Lambda_p := \left(\frac{N-1}{p} \right)^p \quad (2.1)$$

appearing there is optimal. This is in fact a particular case of the work given in [28], but we provide a simple proof below for the convenience of the reader.

Lemma 2.1. *Let $N \geq 2$, $p \geq 2$ and set Λ_p as in (2.1). There holds*

$$\inf_{u \in W^{1,p}(\mathbb{H}^N) \setminus \{0\}} \frac{\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p \, dv_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} |u|^p \, dv_{\mathbb{H}^N}} = \Lambda_p. \quad (2.2)$$

Proof. Considering the upper half space model for \mathbb{H}^N , namely $\mathbb{R}_+^N = \{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}^+\}$ endowed with the Riemannian metric $g_{ij} = \frac{\delta_{ij}}{y^2}$ and using the expression of p -Laplacian (1.3) in these coordinates we have

$$\Delta_{p, \mathbb{H}^N} u = y^N \partial_i (y^{p-N} |\nabla u|^{p-2} \partial_i u).$$

By computing $-\Delta_{p, \mathbb{H}^N}$ for the function $u(x, y) := y^\alpha \in W_{loc}^{1,p}(\mathbb{H}^N)$ where $\alpha > 0$ is a real parameter, one has

$$-\Delta_{p, \mathbb{H}^N} y^\alpha = \alpha^{p-2} \alpha (N-1 - \alpha(p-1)) y^{\alpha(p-1)}.$$

Since

$$\max_{\alpha > 0} [\alpha^{p-2} \alpha(N-1-\alpha(p-1))] = \Lambda_p \quad \text{for } \alpha = \left(\frac{N-1}{p} \right),$$

by the Allegretto-Piepenbrink Theorem for p -Laplacian setting, (for detail see [29, Theorem 2.3]), we readily conclude that

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p \, dv_{\mathbb{H}^N} \geq \Lambda_p \int_{\mathbb{H}^N} |u|^p \, dv_{\mathbb{H}^N}$$

for all $u \in W^{1,p}(\mathbb{H}^N)$. On the other hand, for $\varepsilon > 0$, set

$$U_\varepsilon(x, y) = \left(\frac{y}{(1+y)^2 + |x|^2} \right)^{\frac{N-1+\varepsilon}{p}}.$$

Since in the coordinates (x, y) the volume element reads $dv_{\mathbb{H}^N} = \frac{dx \, dy}{y^N}$ and $\nabla_{\mathbb{H}^N} u = y^2 \nabla u$, we get

$$\int_{\mathbb{H}^N} |U_\varepsilon|^p \, dv_{\mathbb{H}^N} = \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \left(\frac{y}{(1+y)^2 + |x|^2} \right)^{N-1+\varepsilon} \frac{dx \, dy}{y^N}$$

and

$$\begin{aligned} & \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} U_\varepsilon|^p \, dv_{\mathbb{H}^N} \\ &= \left(\frac{N-1+\varepsilon}{p} \right)^p \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \left(\frac{(1-y^2+|x|^2)^2 + 4|x|^2 y^2}{((1+y)^2 + |x|^2)^2} \right)^{p/2} \left(\frac{y}{(1+y)^2 + |x|^2} \right)^{N-1+\varepsilon} \frac{dx \, dy}{y^N} \\ &\leq \left(\frac{N-1+\varepsilon}{p} \right)^p \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \left(\frac{y}{(1+y)^2 + |x|^2} \right)^{N-1+\varepsilon} \frac{dx \, dy}{y^N} \end{aligned}$$

Hence, $U_\varepsilon(x, y) \in W^{1,p}(\mathbb{H}^N)$ for $\varepsilon > 0$ and $\frac{\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} U_\varepsilon|^p \, dv_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} |U_\varepsilon|^p \, dv_{\mathbb{H}^N}} \leq \left(\frac{N-1+\varepsilon}{p} \right)^p$. By letting $\varepsilon \rightarrow 0$, this argument completes the proof of the lemma. \square

Now we are in a situation to state our main results.

In first place, by exploiting the half-space model for \mathbb{H}^N and following the approach of [31], here below we provide a weight that does not globally decay at infinity but which is bounded near x_0 . Hence, this choice turns out to be best suited to capture the non euclidean behaviour of \mathbb{H}^N which occurs at infinity. More precisely, we prove

Theorem 2.2. *Let $p \geq 2$, $N \geq 2$ and set Λ_p as in (2.1). There exists a bounded weight $0 < V \leq 1$ such that for all $u \in C_c^\infty(\mathbb{H}^N)$ there holds*

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p \, dv_{\mathbb{H}^N} - \Lambda_p \int_{\mathbb{H}^N} |u|^p \, dv_{\mathbb{H}^N} \geq \left(\frac{N-1}{p} \right)^{p-1} \frac{1}{4(N-1)} \int_{\mathbb{H}^N} V |u|^p \, dv_{\mathbb{H}^N}. \quad (2.3)$$

Furthermore, set $r := \varrho(x, x_0)$ with $x_0 \in \mathbb{H}^N$ fixed, we have

- for any $0 < \alpha \leq 1$ there exists an unbounded set $U_\alpha \subset \mathbb{H}^N$ such that $V|_{U_\alpha} \equiv \alpha$ and $U_\alpha \cap (B(x_0, 2r) \setminus B(x_0, r)) \neq \emptyset$ as $r \rightarrow +\infty$;
- for any $\beta > 0$ there exists an unbounded set $W_\beta \subset \mathbb{H}^N$ such that $V|_{W_\beta} \sim \sqrt{\frac{\beta}{2}} e^{-r/2}$ as $r \rightarrow +\infty$.

It is worth noticing that the weight V can be written, in the half-space model, as $V(x_1, \dots, x_{N-1}, y) := \frac{y}{\sqrt{y^2 + x_1^2}}$, see Theorem 3.2 in Section 3 from which the above statements follow.

Even if both the inequalities provided by Proposition 1.1 and Theorem 2.2 are of the form (1.4) they seem to lose too much information, respectively, at infinity or near the origin. To this aim, a good compromise is represented by the following Poincaré-Hardy inequality

Theorem 2.3. *Let $p \geq 2$ and $N \geq 1 + p(p-1)$. Set Λ_p as in (2.1) and $r := \varrho(x, x_0)$ with $x_0 \in \mathbb{H}^N$ fixed. Then for $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ there holds*

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p dv_{\mathbb{H}^N} - \Lambda_p \int_{\mathbb{H}^N} |u|^p dv_{\mathbb{H}^N} \geq (p-1) \left(\frac{N-1}{p} \right)^{p-2} \left(\frac{p-1}{p} \right)^2 \int_{\mathbb{H}^N} \frac{|u|^p}{r^p} dv_{\mathbb{H}^N}. \quad (2.4)$$

Remark 2.1. From the above Theorem, we can easily infer that the best constant in the r.h.s. of (2.4), i.e.

$$c_p := \inf_{C_c^\infty(\mathbb{H}^N) \setminus \{0\}} \frac{\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p dv_{\mathbb{H}^N} - \Lambda_p \int_{\mathbb{H}^N} |u|^p dv_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} \frac{|u|^p}{r^p} dv_{\mathbb{H}^N}},$$

blows up as $N \rightarrow \infty$ if $p > 2$. This does not happen in the linear case $p = 2$, where $c_2 = \frac{1}{4}$, see (1.1), where it is known that the constant c_2 is optimal. This issue was proved in [6] by providing an explicit super-solution for the corresponding Euler-equation, a construction that also allows to determine a remainder term for (1.1) of the type $\frac{1}{\sinh^2 r}$, see Remark 7.1. Unfortunately, this argument carries over to the case $p > 2$ only partially thereby allowing to prove Theorem 2.4 below on suitable geodesic balls.

Remark 2.2. In Theorem 2.3, the restriction $N \geq 1 + p(p-1)$ is technical and only comes from the last step in the proof. Nevertheless, the very same assumption also appears in the Poincaré-Hardy inequality below involving the same weight but holding on geodesic balls. Since the proofs of the two theorems are completely different, we are led to believe that a deeper relation between the dimension restriction and the weight considered might exist.

Theorem 2.4. *Let $p \geq 2$ and $N \geq 1 + p(p-1)$. Let Λ_p be as in (2.1) and $r := \varrho(x, x_0)$ with $x_0 \in \mathbb{H}^N$ fixed. Then for $u \in C_c^\infty(B(x_0, r_p))$ there holds*

$$\begin{aligned} & \int_{B(x_0, r_p)} |\nabla_{\mathbb{H}^N} u|^p dv_{\mathbb{H}^N} - \Lambda_p \int_{B(x_0, r_p)} |u|^p dv_{\mathbb{H}^N} \\ & \geq \frac{(p-1)^{p-1} (N(p-2) + 1)}{p^p} \int_{B(x_0, r_p)} \frac{|u|^p}{r^p} dv_{\mathbb{H}^N} \\ & \quad + \frac{(N-1)(N-1-p(p-1))(p-1)^{p-2}}{p^p} \int_{B(x_0, r_p)} \frac{|u|^p}{\sinh^p r} dv_{\mathbb{H}^N} \end{aligned} \quad (2.5)$$

where $B(x_0, r_p)$ is the geodesic ball of radius r_p centered at x_0 and where we let, for $p > 2$, $r_p = r_p(N)$ be the unique positive solution to the equation

$$\coth r_p - 1 - \frac{p-1}{N-1} \frac{1}{r_p} = 0,$$

whereas $r_2 := +\infty$ (namely $B(x_0, r_2) = \mathbb{H}^N$).

In particular, for every $p > 2$ the map $N \mapsto r_p(N)$ is strictly increasing in $[1 + p(p - 1), +\infty)$ and $\lim_{N \rightarrow +\infty} r_p(N) = +\infty$ while, for every $N > 3$ the map $p \mapsto r_p$ is strictly decreasing in $(2, \frac{1+\sqrt{4N-3}}{2}]$.

The proof of the above Theorem will be an elementary corollary of Theorem 7.1 below, which states an inequality like (2.5) but valid on the whole \mathbb{H}^N and in which the constant Λ_p is replaced by a non-constant weight: $\Lambda_p H_p(r)$. Here, $H_p(r)$ is a positive function which is larger than one in $(0, r_p)$, smaller than one in $(r_p, +\infty)$, and that converges to one as $r \rightarrow +\infty$, see Figure 1 in Section 7. Hence, such inequality is not an improvement of the p -Poincaré inequality if $p \neq 2$, but has anyway an independent interest in itself.

3. RELATED HARDY-MAZ'YA-TYPE INEQUALITIES ON HALF-SPACE

This section is devoted to the study of improved Hardy-Maz'ya-type inequalities on upper half space. There have been an extensive research on Hardy-Maz'ya inequality (see [17, 19, 24, 26]). Our main goal here is to present some Hardy-Maz'ya inequalities strictly related to our Poincaré-Hardy inequalities on the hyperbolic space. We begin with the counterpart of Lemma 2.1:

Lemma 3.1. *Let $p \geq 2$, $N \geq 2$ and set Λ_p as in (2.1). Then for all $u \in C_c^\infty(\mathbb{R}_+^N)$ there holds*

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|\nabla u|^p}{y^{N-p}} dx dy \geq \Lambda_p \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|u|^p}{y^N} dx dy, \quad (3.1)$$

where ∇u denotes the euclidean gradient. Moreover the constant Λ_p appearing in (3.1) is sharp.

Proof. The proof of Lemma 3.1 follows by noticing that in the upper half space model for \mathbb{H}^N , see the proof of Lemma 2.1, (2.2) readily writes as the Hardy-Maz'ya-type inequality (3.1). Hence, the statement of Lemma 3.1 comes as a corollary of Lemma 2.1. \square

Next we turn to the main result of this section. We improve (3.1) by providing a suitable remainder term.

Theorem 3.2. *Let $p \geq 2$, $N \geq 2$ and set Λ_p as in (2.1). For all $u \in C_c^\infty(\mathbb{R}_+^N)$ there holds*

$$\begin{aligned} & \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|\nabla u|^p}{y^{N-p}} dx dy - \Lambda_p \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|u|^p}{y^N} dx dy \geq \\ & \left(\frac{N-1}{p} \right)^{p-1} \frac{1}{4(N-1)} \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|u|^p}{y^{N-1} \sqrt{y^2 + x_1^2}} dx dy. \end{aligned} \quad (3.2)$$

It's worth noting that Theorem 2.2 turns out to be a consequence of the above theorem. We postpone the proofs of Theorem 3.2 and, hence, of Theorem 2.2 to Section 5.

Remark 3.1. Let $0 < \tau \leq 1$. Following [31, Theorem 5.1], the same proof of Theorem 3.2 with minor changes yields the following variant of (3.2):

$$\begin{aligned} & \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|\nabla u|^p}{y^{N-p}} dx dy - \Lambda_p \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|u|^p}{y^N} dx dy \geq \\ & \left(\frac{N-1}{p} \right)^{p-1} \frac{1}{4(N-1)} \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|u|^p}{y^{N-\tau} (y^2 + x_1^2)^{\tau/2}} dx dy, \end{aligned}$$

for all $u \in C_c^\infty(\mathbb{R}_+^N)$.

4. PROOF OF PROPOSITION 1.1

We recall for the convenience of the reader the proof given in [7], only the asymptotics at infinity not being explicitly given there. The proof relies on the well known classical Hardy inequality with respect to the Green's function and exploiting its behavior on hyperbolic space. More precisely, for $N \geq 2$ and $p \geq 2$, we recall the following Hardy inequality [7, Proposition 4.4]:

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p \, dv_{\mathbb{H}^N} \geq \left(\frac{p-1}{p} \right)^p \int_{\mathbb{H}^N} \left| \frac{\nabla G_p}{G_p} \right|^p |u|^p \, dv_{\mathbb{H}^N}, \quad (4.1)$$

for $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$, where G_p is the Green's function of $-\Delta_{p,\mathbb{H}^N}$ and is given by

$$G_p(r) := c \int_r^\infty (\sinh s)^{-\frac{N-1}{p-1}} \, ds$$

for some suitable constant $c > 0$.

The proof is then a calculus exercise involving the asymptotics of the function $G_p(r)$. Indeed, Eq. (4.1) may be rewritten as

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p \, dv_{\mathbb{H}^N} - \Lambda_p \int_{\mathbb{H}^N} |u|^p \, dv_{\mathbb{H}^N} \geq \int_{\mathbb{H}^N} W |u|^p \, dv_{\mathbb{H}^N},$$

where

$$W(r) := \left(\frac{p-1}{p} \right)^p \left| \frac{G_p'(r)}{G_p(r)} \right|^p - \Lambda_p,$$

with Λ_p as in (2.1).

First we claim that $W > 0$. From the expression of the Green's function we have

$$\begin{aligned} G_p(r) &= \int_r^\infty (\sinh s)^{-\frac{N-1}{p-1}} \, ds = \int_r^\infty (\sinh s)^{-\frac{N-1}{p-1}-1} \sinh s \, ds \\ &< \int_r^\infty (\sinh s)^{-\frac{N-1}{p-1}-1} \cosh s \, ds = \int_{\sinh r}^\infty t^{-\frac{N-1}{p-1}-1} \, dt \\ &= \frac{p-1}{N-1} (\sinh r)^{-\frac{N-1}{p-1}}. \end{aligned}$$

Moreover, we also have $G_p'(r) = -(\sinh r)^{-\frac{N-1}{p-1}}$. Therefore,

$$\left| \frac{G_p'(r)}{G_p(r)} \right|^p > \left(\frac{N-1}{p-1} \right)^p,$$

and hence this proves $\left(\frac{p-1}{p} \right)^p \left| \frac{G_p'(r)}{G_p(r)} \right|^p > \Lambda_p$.

Let us turn to study the asymptotic behavior of W near the origin. First consider the case when $N \geq p$. Then, $G_p(r) \rightarrow \infty$ as $r \rightarrow 0$ and, using de L'Hôpital's rule, we obtain:

$$\lim_{r \rightarrow 0} \frac{r G_p'(r)}{G_p(r)} = \frac{p-N}{p-1} \quad \text{if } N > p$$

and

$$\lim_{r \rightarrow 0} \frac{r \log r G_p'(r)}{G_p(r)} = 1 \quad \text{if } N = p.$$

Whence, the stated asymptotics easily follows.

When $N < p$, in the second term above one has $\int_r^\infty (\sinh s)^{-\frac{N-1}{p-1}} \, ds < \infty$ as $r \rightarrow 0$. Hence, (1.5) follows immediately by exploiting $\sinh r \sim r$ as $r \rightarrow 0$.

Finally, we study the asymptotics of W near infinity. For this we note that

$$\begin{aligned} G_p(r) &= \int_r^\infty (\sinh s)^{-\frac{N-1}{p-1}} ds = \int_{\sinh r}^\infty t^{-\frac{N-1}{p-1}} (1+t^2)^{-\frac{1}{2}} dt \\ &= \int_{\sinh r}^\infty t^{-\frac{N-1}{p-1}-1} \left[1 - \frac{1}{2t^2} + o\left(\frac{1}{t^2}\right) \right] dt, \quad r \rightarrow \infty \\ &= \frac{p-1}{N-1} (\sinh r)^{-\frac{N-1}{p-1}} - C_{N,p} (\sinh r)^{-\frac{N-1}{p-1}-2} + o\left((\sinh r)^{-\frac{N-1}{p-1}-2}\right), \end{aligned}$$

hence we have

$$\left| \frac{G'_p(r)}{G_p(r)} \right|^p = \left(\frac{N-1}{p-1} \right)^p + \tilde{A}_N (\sinh r)^{-2} + o((\sinh r)^{-2}).$$

This completes the proof.

5. PROOF OF THEOREM 3.2 AND THEOREM 2.2

Proof of Theorem 3.2

The key ingredient in the proof is the following lemma from [31] that we adapt to our situation with a suitable choice of the parameters.

Lemma 5.1. [31, Lemma 2.1] *Let Ω be a convex domain in \mathbb{R}^N and set $\delta(z) := \text{dist}(z, \partial\Omega)$ for any $z \in \Omega$. Let $d \in (-\infty, mp-1)$ where $m \in \mathbb{N}_+$ and let $\mathbf{F} = (F_1, \dots, F_N)$ be a $C^1(\Omega)$ vector field in \mathbb{R}^N . Furthermore, let $w \in C^1(\Omega)$ be a nonnegative weight function and*

$$h_{p,m,d} := \left(\frac{mp-d-1}{p} \right)^p.$$

Then, the following inequality holds

$$\begin{aligned} \int_\Omega \frac{|\nabla u|^p w}{\delta^{(m-1)p-d}} dz &\geq h_{p,m,d} \left(\int_\Omega \frac{|u|^p w}{\delta^{mp-d}} - \frac{p|u|^p \Delta \delta w}{(mp-d-1)\delta^{mp-d-1}} dz \right) \\ &+ h_{p,m,d} \int_\Omega \left[\frac{p \operatorname{div} \mathbf{F}}{mp-d-1} + \frac{p-1}{\delta^{mp-d}} \left(1 - |\nabla \delta - \delta^{mp-d-1} \mathbf{F}|^{\frac{p}{p-1}} \right) \right] |u|^p w dz \\ &+ \left(\frac{mp-d-1}{p} \right)^{p-1} \int_\Omega \nabla w \cdot \left(\mathbf{F} - \frac{\nabla \delta}{\delta^{mp-d-1}} \right) |u|^p dz, \end{aligned} \quad (5.1)$$

for all $u \in C_c^\infty(\Omega)$.

We will apply Lemma 5.1 with $\Omega = \mathbb{R}_+^N$. Hence, $z = (x_1, \dots, x_{N-1}, y) = (x, y)$ with $x \in \mathbb{R}^{N-1}$, $y \in \mathbb{R}^+$, and $\delta(z) = y$. Furthermore, we fix $w = 1$, $m = 2$ and $d = mp - N$ so that $d < mp - 1$ for any $p \geq 2$ and $N \geq 2$ and we obtain $h_{p,m,d} = \Lambda_p$. Then, (5.1) reads as follows.

Lemma 5.2. *Let $p \geq 2$, $N \geq 2$ and set Λ_p as in (2.1). For any $C^1(\mathbb{R}_+^N)$ vector field $\mathbf{F} = (F_1, \dots, F_N)$, the following inequality holds*

$$\begin{aligned} \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|\nabla u|^p}{y^{N-p}} dx dy - \Lambda_p \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|u|^p}{y^N} dx dy &\geq \\ \Lambda_p \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \left[\frac{p \operatorname{div} \mathbf{F}}{N-1} + \frac{p-1}{y^N} \left(1 - |(0, \dots, 0, 1) - y^{N-1} \mathbf{F}|^{\frac{p}{p-1}} \right) \right] |u|^p dx dy, \end{aligned} \quad (5.2)$$

for all $u \in C_c^\infty(\mathbb{R}_+^N)$.

Next, in the spirit of [31, Theorem 4.1], for any $0 \leq a \leq 1$ we write (5.2) with $\mathbf{F} = \left(0, \dots, \frac{a}{y^{N-2}\sqrt{y^2+x_1^2}}\right)$. Since

$$\operatorname{div} \mathbf{F} \geq -\frac{a(N-2)}{y^{N-1}\sqrt{y^2+x_1^2}} - \frac{a}{y^{N-2}(y^2+x_1^2)}$$

and, by using $(1+s)^b \leq 1+bs + (b/2)s^2$ for $b \leq 2$ and $s \geq -1$, we have

$$\frac{p-1}{y^N} \left(1 - (1 - y^{N-1}F_N)^{\frac{p}{p-1}}\right) \geq \frac{1}{y^N} \left(\frac{pay}{\sqrt{y^2+x_1^2}} - \frac{pa^2y^2}{2(y^2+x_1^2)}\right).$$

Therefore, from (5.2) we obtain

$$\begin{aligned} & \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|\nabla u|^p}{y^{N-p}} \, dx \, dy - \Lambda_p \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|u|^p}{y^N} \, dx \, dy \geq \\ & \quad a \left(\frac{N-1}{p}\right)^{p-1} \int_{\mathbb{R}^{N-1}} \frac{|u|^p}{y^{N-1}\sqrt{y^2+x_1^2}} \, dx \, dy \\ & \quad - \left(a + \frac{(N-1)a^2}{2}\right) \left(\frac{N-1}{p}\right)^{p-1} \int_{\mathbb{R}^{N-1}} \frac{|u|^p}{y^{N-2}(y^2+x_1^2)} \, dx \, dy \end{aligned} \quad (5.3)$$

for all $u \in C_c^\infty(\mathbb{R}_+^N)$.

Similarly, for any $0 \leq c \leq 1$, writing (5.2) with $\mathbf{F} = c \left(\frac{x_1 y^{2-N}}{(y^2+x_1^2)}, 0, \dots, 0, \frac{y^{3-N}}{(y^2+x_1^2)}\right)$ and maximizing in c , for $c = 1/(N-1)$ we get

$$\begin{aligned} & \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|\nabla u|^p}{y^{N-p}} \, dx \, dy - \Lambda_p \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|u|^p}{y^N} \, dx \, dy \\ & \geq \left(\frac{N-1}{p}\right)^{p-1} \frac{1}{2(N-1)} \int_{\mathbb{R}^{N-1}} \frac{|u|^p}{y^{N-2}(y^2+x_1^2)} \, dx \, dy, \end{aligned} \quad (5.4)$$

for all $u \in C_c^\infty(\mathbb{R}_+^N)$. Finally, adding (5.3) to (5.4) multiplied by $2(N-1) \left(a + \frac{(N-1)a^2}{2}\right)$ and maximizing in a , for $a = 1/(N-1)$ we finally obtain (3.2). This concludes the proof of Theorem 3.2.

Proof of Theorem 2.2

Letting $V(x_1, \dots, x_{N-1}, y) := \frac{y}{\sqrt{y^2+x_1^2}}$, the proof of (2.3) follows at once from (3.2) by exploiting the half-space model for \mathbb{H}^N as explained in the proof of Lemma 2.1. Next, for any $\alpha \in (0, 1]$, set $U_\alpha := \{(x, y) \in \mathbb{R}_+^N : x_1 = ky \text{ with } k^2 = (1-\alpha^2)/\alpha^2\}$. Clearly, $V|_{U_\alpha} \equiv \alpha$ and $V|_{U_\alpha} \rightarrow \alpha$ as $y \rightarrow +\infty$. Set $r := \varrho((x, y), (0, 1))$. Since $\cosh(r(x, y)) = \left(1 + \frac{(y-1)^2+|x|^2}{2y}\right)$, we get that $r(x, y) \rightarrow +\infty$ as $y \rightarrow +\infty$ and the corresponding claim of Theorem 2.2 follows.

On the other hand, for any $\beta > 0$, take $W_\beta := \{(x_1, 0, \dots, 0, \beta) \in \mathbb{R}_+^N\}$. Then, for any $\beta > 0$, one has $V|_{W_\beta} \rightarrow 0$ as $x_1 \rightarrow +\infty$. Furthermore, $r|_{W_\beta} \rightarrow +\infty$ if and only if $x_1 \rightarrow +\infty$ and $V|_{W_\beta} \sim \sqrt{\frac{\beta}{2}} e^{-r/2}$ as $r \rightarrow +\infty$.

6. PROOF OF THEOREM 2.3

Before proving Theorem 2.3, we recall some known results related to the symmetrization on the hyperbolic space. For any $\Omega \subset \mathbb{H}^N$ and $x_0 \in \mathbb{H}^N$ fixed, denote with Ω^* the geodesic ball $B(x_0, r)$ having the same measure of Ω . For $u \in C_c^\infty(\Omega)$, the hyperbolic symmetrization of u is the unique nonnegative and decreasing function u^* defined in Ω^* such that the level sets $\{x \in \Omega^* : u^*(x) > t\}$ are concentric balls having the same measure of the level sets $\{x \in \Omega : |u(x)| > t\}$. See [2] for more details.

Lemma 6.1. *Let $p \geq 1$ and $N \geq 2$. For every $u, v \in C_c^\infty(\mathbb{H}^N)$, there holds*

$$\begin{aligned} \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p \, dv_{\mathbb{H}^N} &\geq \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u^*|^p \, dv_{\mathbb{H}^N}, \\ \int_{\mathbb{H}^N} |u|^p \, dv_{\mathbb{H}^N} &= \int_{\mathbb{H}^N} |u^*|^p \, dv_{\mathbb{H}^N}, \end{aligned}$$

and

$$\int_{\mathbb{H}^N} |uv| \, dv_{\mathbb{H}^N} \leq \int_{\mathbb{H}^N} u^* v^* \, dv_{\mathbb{H}^N},$$

where $*$ denotes the hyperbolic symmetrization.

Next we state a p -convexity lemma. The proof of the following lemma can be obtained as an application of Taylor's formula, we refer to [20] for further details.

Lemma 6.2. *Let $p \geq 1$ and ξ, η be real numbers such that $\xi \geq 0$ and $\xi - \eta \geq 0$. Then*

$$(\xi - \eta)^p + p\xi^{p-1}\eta - \xi^p \geq \begin{cases} \max\{(p-1)\eta^2\xi^{p-2}, |\eta|^p\}, & \text{if } p \geq 2, \\ \frac{1}{2}p(p-1)\frac{\eta^2}{(\xi+|\eta|)^{2-p}}, & \text{if } 1 \leq p \leq 2. \end{cases}$$

Now we turn to prove an *optimal* inequality which is one of the key ingredient in proving Theorem 2.3.

Lemma 6.3. *For all $v \in C_c^\infty(0, \infty)$ and $2 \leq l \leq p$, there holds*

$$\int_0^\infty |v(r)|^{p-l} (\coth r)^{p-l} |v'(r)|^l \, dr \geq \left(\frac{p-1}{p}\right)^l \int_0^\infty \frac{|v(r)|^p}{r^p} \, dr. \quad (6.1)$$

Furthermore, the constant $\left(\frac{p-1}{p}\right)^l$ in (6.1) is sharp.

Proof. Write

$$\begin{aligned} \int_0^\infty \frac{|v(r)|^p}{r^p} \, dr &= \frac{-1}{p-1} \int_0^\infty |v(r)|^p \frac{d}{dr} (r^{-(p-1)}) \, dr \\ &= \left(\frac{p}{p-1}\right) \int_0^\infty \frac{|v(r)|^{p-2} v(r) v'(r)}{r^{p-1}} \, dr \\ &\leq \left(\frac{p}{p-1}\right) \int_0^\infty \frac{|v(r)|^{p-1} |v'(r)|}{r^{p-1}} \, dr \\ &= \left(\frac{p}{p-1}\right) \int_0^\infty \frac{|v(r)|^{\frac{p(l-1)}{l}}}{r^{\frac{p(l-1)}{l}}} \frac{|v(r)|^{\frac{p-l}{l}} |v'(r)|}{r^{\frac{p-l}{l}}} \, dr \\ &\leq \left(\frac{p}{p-1}\right) \left(\int_0^\infty \frac{|v(r)|^p}{r^p} \, dr \right)^{\frac{l-1}{l}} \left(\int_0^\infty \frac{|v(r)|^{p-l} |v'(r)|^l}{r^{p-l}} \, dr \right)^{\frac{1}{l}}. \end{aligned}$$

Since $\coth r \geq \frac{1}{r}$ for all $r > 0$, we conclude

$$\int_0^\infty |v(r)|^{p-l} (\coth r)^{p-l} |v'(r)|^l dr \geq \left(\frac{p-1}{p} \right)^l \int_0^\infty \frac{|v(r)|^p}{r^p} dr.$$

Next we turn to the optimality issue. For $\varepsilon > 0$ and $\delta > 0$, consider

$$V_\varepsilon^\delta(r) := \begin{cases} r^{\frac{p-1+\delta}{p}}, & 0 < r < \varepsilon \\ \varepsilon^{\frac{p-1+\delta}{p}}, & \varepsilon \leq r < 1 \\ \varepsilon^{\frac{p-1+\delta}{p}} (2-r), & 1 \leq r < 2 \\ 0, & r \geq 2. \end{cases}$$

Clearly, $V_\varepsilon^\delta(r) \in W^{1,p}(0, \infty)$ for $\varepsilon > 0, \delta > 0$. Furthermore, we have

$$\int_0^\infty \frac{|V_\varepsilon^\delta(r)|^p}{r^p} dr \geq \int_0^\varepsilon \frac{r^{p-1+\delta}}{r^p} dr = \int_0^\varepsilon r^{\delta-1} dr.$$

On the other hand, using the fact $\sinh r \geq r$, we obtain

$$\begin{aligned} & \int_0^\infty |V_\varepsilon^\delta(r)|^{p-l} (\coth r)^{p-l} |(V_\varepsilon^\delta(r))'|^l dr = \\ & \left(\frac{p-1+\delta}{p} \right)^l \int_0^\varepsilon r^{\frac{(p-1+\delta)(p-l)}{p}} (\coth r)^{p-l} r^{\frac{(\delta-1)l}{p}} dr \\ & + \varepsilon^{p-1+\delta} \int_1^2 (2-r)^{p-l} (\coth r)^{p-l} dr \\ & = \left(\frac{p-1+\delta}{p} \right)^l \int_0^\varepsilon r^{p-1+\delta-l} (\coth r)^{p-l} dr + c\varepsilon^{p-1+\delta} \\ & \leq \left(\frac{p-1+\delta}{p} \right)^l (\cosh \varepsilon)^{p-l} \int_0^\varepsilon \frac{r^{p-1+\delta-l}}{(\sinh r)^{p-l}} dr + c\varepsilon^{p-1+\delta} \\ & \leq \left(\frac{p-1+\delta}{p} \right)^l (\cosh \varepsilon)^{p-l} \int_0^\varepsilon r^{\delta-1} dr + c\varepsilon^{p-1+\delta}. \end{aligned}$$

Hence,

$$Q := \inf_{v \in W^{1,p}(0,\infty) \setminus \{0\}} \frac{\int_0^\infty |v(r)|^{p-l} (\coth r)^{p-l} |v'(r)|^l dr}{\int_0^\infty \frac{|v(r)|^p}{r^p} dr} \leq \left(\frac{p-1+\delta}{p} \right)^l (\cosh \varepsilon)^{p-l} + c\delta\varepsilon^{p-1}.$$

First letting $\varepsilon \rightarrow 0$, and then with $\delta \rightarrow 0$, we conclude that

$$Q \leq \left(\frac{p-1}{p} \right)^l.$$

This proves the optimality and concludes the proof. \square

Proof of Theorem 2.3.

By hyperbolic symmetrization, i.e., in view of Lemma 6.1, we may assume $u \in C_c^\infty(\mathbb{H}^N)$ nonnegative, radially symmetric and non increasing. Hence, to prove (2.4), it is enough to show the validity of the following inequality

$$\begin{aligned}
& \int_0^\infty |u'(r)|^p (\sinh r)^{N-1} \, dr - \left(\frac{N-1}{p} \right)^p \int_0^\infty (u(r))^p (\sinh r)^{N-1} \, dr \\
& \geq (p-1) \left(\frac{N-1}{p} \right)^{p-2} \left(\frac{p-1}{p} \right)^2 \int_0^\infty \frac{(u(r))^p}{r^p} (\sinh r)^{N-1} \, dr.
\end{aligned} \tag{6.2}$$

Let us define a suitable transformation which allows to put the Poincaré term into evidence:

$$v(r) := (\sinh r)^{\frac{N-1}{p}} u(r)$$

so that

$$v'(r) = (u'(r))(\sinh r)^{\frac{N-1}{p}} + \left(\frac{N-1}{p} (\sinh r)^{\frac{N-1}{p}} \coth r \right) u,$$

and

$$(u'(r))(\sinh r)^{\frac{N-1}{p}} = v'(r) - \left(\frac{N-1}{p} (\sinh r)^{\frac{N-1}{p}} \coth r \right) u.$$

At this point we apply the p -convexity Lemma 6.2. By taking

$$\xi = \left(\frac{N-1}{p} \right) (\sinh r)^{\frac{N-1}{p}} \coth r u > 0 \quad \text{and} \quad \eta = v'(r)$$

and using Lemma 6.2 for $p \geq 2$, we obtain

$$\begin{aligned}
|u'(r)|^p (\sinh r)^{N-1} & \geq (p-1) \left(\frac{N-1}{p} \right)^{p-2} v^{p-2}(r) (\coth r)^{p-2} (v'(r))^2 \\
& + \left(\frac{N-1}{p} \right)^p (\sinh r)^{N-1} (\coth r)^p u^p(r) \\
& - p \left(\frac{N-1}{p} \right)^{p-1} (\sinh r)^{\frac{(N-1)(p-1)}{p}} (\coth r)^{p-1} u^{p-1}(r) v'(r) \\
& = (p-1) \left(\frac{N-1}{p} \right)^{p-2} v^{p-2}(r) (\coth r)^{p-2} (v'(r))^2 \\
& + \left(\frac{N-1}{p} \right)^p (\sinh r)^{N-1} (\coth r)^p u^p(r) \\
& - p \left(\frac{N-1}{p} \right)^{p-1} (\coth r)^{p-1} v^{p-1}(r) v'(r).
\end{aligned}$$

Integrating both sides of above inequality and applying Lemma 6.3 with $l = 2$, we get

$$\begin{aligned}
\int_0^\infty |u'(r)|^p (\sinh r)^{N-1} \, dr & \geq (p-1) \left(\frac{N-1}{p} \right)^{p-2} \int_0^\infty v^{p-2}(r) (\coth r)^{p-2} (v'(r))^2 \, dr \\
& + \left(\frac{N-1}{p} \right)^p \int_0^\infty (\coth r)^p v^p(r) \, dr
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{N-1}{p} \right)^{p-1} \int_0^\infty (\coth r)^{p-1} \frac{d}{dr} (v(r))^p \, dr \\
& \geq (p-1) \left(\frac{N-1}{p} \right)^{p-2} \left(\frac{p-1}{p} \right)^2 \int_0^\infty \frac{v^p(r)}{r^p} \, dr \\
& + \left(\frac{N-1}{p} \right)^p \int_0^\infty F(r) (v(r))^p \, dr,
\end{aligned}$$

where $F(r) := (\coth r)^p - \frac{p(p-1)}{N-1} \frac{(\coth r)^p}{\cosh^2 r}$. Then, (6.2) follows by showing that $F(r) \geq 1$ for all $r > 0$ or equivalently that

$$\tilde{F}(r) := (N-1) \cosh^p r - (N-1) \sinh^p r - p(p-1) \cosh^{p-2} r \geq 0,$$

for all $r > 0$. By rewriting

$$\tilde{F}(r) = \cosh^{p-2} r (N-1 - p(p-1)) + (N-1) \sinh^2 r (\cosh^{p-2} r - \sinh^{p-2} r),$$

we immediately infer that $\tilde{F}(r)$ is non negative provided that $N \geq 1 + p(p-1)$, and also the condition is necessary. This completes the proof of Theorem 2.3.

7. PROOF OF THEOREM 2.4

The proof of Theorem 2.4 comes as a corollary of the following weighted inequality:

Theorem 7.1. *Let $p \geq 2$ and $N \geq 1 + p(p-1)$. Set Λ_p as in (2.1) and $r := \varrho(x, x_0)$ with $x_0 \in \mathbb{H}^N$ fixed. Then for $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ there holds*

$$\begin{aligned}
& \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p \, dv_{\mathbb{H}^N} - \Lambda_p \int_{\mathbb{H}^N} H_p(r) |u|^p \, dv_{\mathbb{H}^N} \geq \\
& \frac{(p-1)^{p-1} (N(p-2) + 1)}{p^p} \int_{\mathbb{H}^N} \frac{|u|^p}{r^p} \, dv_{\mathbb{H}^N} \\
& + \frac{(N-1)(N-1-p(p-1))(p-1)^{p-2}}{p^p} \int_{\mathbb{H}^N} \frac{|u|^p}{\sinh^p r} \, dv_{\mathbb{H}^N}
\end{aligned} \tag{7.1}$$

where $H_p(r) = \left(\coth r - \left(\frac{p-1}{N-1} \right) \frac{1}{r} \right)^{p-2}$.

Remark 7.1. When $p = 2$, the statement of Theorem 7.1 includes that of Theorem 2.3 providing a further remainder term.

Before proving Theorem 7.1 we collect here below the main properties of the weight H_p . This will clarify the meaning of inequality (2.5), see also Figure 1.

Lemma 7.2. *Let $H_p : \mathbb{R}^+ \rightarrow \mathbb{R}$ be defined as in the statement of Theorem 7.1 with $p > 2$ and $N \geq 1 + p(p-1)$. Then, the following holds*

- (a) *For all $r > 0$, $H_p(r) > 0$, $H_p(r) \sim \left(\frac{N-p}{N-1} \right)^{p-2} \frac{1}{r^{p-2}}$ as $r \rightarrow 0^+$, and $H_p(r) \rightarrow 1^-$ as $r \rightarrow \infty$.*
- (b) *There exists a unique $r_p \in (0, \infty)$ such that $H_p(r) \geq 1$ for $r \in (0, r_p]$ and $H_p(r) < 1$ for $r \in (r_p, \infty)$.*

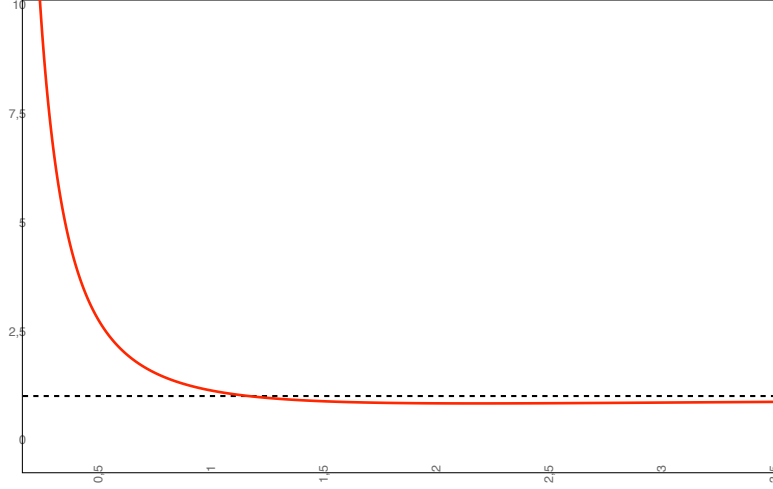


FIGURE 1. The plot of $y = H_p(r)$ for $p = 4$ and $N = 13$. The dotted line is $y = 1$ and the intersection point of the two curves is the point r_p as defined in Lemma 7.2-(b).

Proof. We set

$$\tilde{H}_p(r) := \coth r - \left(\frac{p-1}{N-1} \right) \frac{1}{r}, \quad r > 0.$$

Then, the property of H_p can be readily deduced from that of \tilde{H}_p .

The sign and the asymptotics of \tilde{H}_p follows from fact that

$$\coth r > \frac{1}{r} \text{ in } (0, \infty), \quad \coth r \sim \frac{1}{r} \text{ as } r \rightarrow 0^+, \quad \text{and } \coth r \rightarrow 1 \text{ as } r \rightarrow \infty.$$

To prove assertion (b), we note that

$$\tilde{H}'_p(r) = (N-1)^{-1} \left(\frac{-(N-1)r^2 + (p-1)\sinh^2 r}{r^2 \sinh^2 r} \right) =: \frac{(N-1)^{-1}}{r^2 \sinh^2 r} h(r). \quad (7.2)$$

Since $h'''(r) = 8(p-1)\cosh r \sinh r > 0$ for all $r > 0$, $h''(0) = -2(N-p)$, and $h'(0) = h(0) = 0$ one readily deduces the existence of a unique $r_0 > 0$ such that $h(r) < 0$ in $(0, r_0)$, $h(r_0) = 0$ and $h(r) > 0$ in (r_0, ∞) . Hence, $\tilde{H}'_p(r) < 0$ in $(0, r_0)$ and $\tilde{H}'_p(r) > 0$ in (r_0, ∞) . This fact and assertion (a) gives the existence of a unique $r_p \in (0, r_0)$ for which (b) holds where r_p clearly satisfies

$$\coth r_p - 1 - \frac{p-1}{N-1} \frac{1}{r_p} = 0. \quad (7.3)$$

□

Proof of Theorem 2.4

The proof readily follows by combining the statements of Theorem 7.1 and Lemma 7.2. In particular equation (7.3) implicitly defines a map $N \mapsto r_p(N)$. By differentiating in (7.3) one gets

$$\frac{d}{dN}(r_p(N)) = -\frac{(p-1)r_p \sinh^2 r_p}{(N-1)h(r_p)},$$

where the function h is as defined in (7.2). Since from the proof of Lemma 7.2-(b) we know that $h(r_p) < 0$, we conclude that the map $N \mapsto r_p(N)$ is strictly increasing. On the other hand, equation (7.3) also implicitly defines a map $p \mapsto r_p$. In this case we get

$$\frac{d}{dp}(r_p) = \frac{r_p \sinh^2 r_p}{(N-1)h(r_p)} < 0.$$

Hence, the map $p \mapsto r_p(N)$ is strictly decreasing.

Proof of Theorem 7.1

The p -Laplacian operator in radial coordinates on the hyperbolic space writes

$$\begin{aligned} \Delta_{p, \mathbb{H}^N} u(r) &:= \Delta_p u(r) = (p-1)|u'(r)|^{p-2}u''(r) + (N-1)\coth r|u'(r)|^{p-2}u'(r) \\ &:= |u'(r)|^{p-2}L_p u(r). \end{aligned} \quad (7.4)$$

Set $g(r) = \left(\frac{r}{\sinh r}\right)^{\frac{(N-1)}{p}}$ and $f(r) = r^{\frac{p-N}{p}}$, some straightforward computations give

$$\begin{aligned} L_p g(r) &= \frac{-(N-1)}{p} \left[\frac{(N-1) - p(p-1)}{p} \frac{1}{\sinh^2 r} + \left(\frac{N-1}{p}\right) \right. \\ &\quad \left. + \frac{(p-1)(p-(N-1))}{p} \frac{1}{r^2} + \frac{(N-1)(p-2)}{p} \frac{1}{r} \right], \end{aligned} \quad (7.5)$$

and

$$L_p f(r) = \left[\frac{N(N-p)(p-1)}{p^2} \frac{1}{r^2} - (N-1)\coth r \frac{N-p}{p} \frac{1}{r} \right] f(r) \quad (7.6)$$

Using (7.5) and (7.6), we deduce for $\tilde{g}(r) = g(r)f(r)$,

$$\begin{aligned} L_p \tilde{g}(r) &= (L_p g(r))f(r) + (L_p f(r))g(r) \\ &\quad + 2(p-1) \left(\frac{-(N-1)}{p} \coth r + \frac{N-1}{p} \frac{1}{r} \right) g(r)f'(r) \\ &= \left(\frac{N-1}{p}\right)^2 \tilde{g} + \frac{(p-1)^2}{p^2} \frac{1}{r^2} \tilde{g} + \frac{(p-1)(p-2)(N-1)}{p^2} \left(\frac{\coth r}{r}\right) \tilde{g} \\ &\quad + \frac{(N-1)(N-1-p(p-1))}{p^2} \frac{1}{\sinh^2 r} \tilde{g}. \end{aligned} \quad (7.7)$$

In view of Eq. (7.4) and Eq. (7.7) we obtain

$$\begin{aligned} -\Delta_p \tilde{g} - \left(\frac{N-1}{p}\right)^2 |\tilde{g}'|^{p-2} \tilde{g} &= \\ \frac{(p-1)^2}{p^2} \frac{1}{r^2} |\tilde{g}'|^{p-2} \tilde{g} + \frac{(p-1)(p-2)(N-1)}{p^2} \left(\frac{\coth r}{r}\right) |\tilde{g}'|^{p-2} \tilde{g} \\ &\quad + \frac{(N-1)(N-1-p(p-1))}{p^2} \frac{1}{\sinh^2 r} |\tilde{g}'|^{p-2} \tilde{g}. \end{aligned} \quad (7.8)$$

Furthermore, we have

$$\begin{aligned}\tilde{g}'(r) &= (g'(r))f(r) + (f'(r))g(r) \\ &= -\frac{1}{p} \left((N-1) \coth r - (p-1) \frac{1}{r} \right) \tilde{g}(r).\end{aligned}\tag{7.9}$$

Namely,

$$|\tilde{g}'(r)|^{p-2} = \left(\frac{N-1}{p} \right)^{p-2} H_p(r) \tilde{g}^{p-2}(r),$$

with $H_p(r)$ as defined in the statement of Theorem 2.4. On the other hand, a further computation using (7.9) and the fact $\coth r > \frac{1}{r}$, gives

$$\begin{aligned}|\tilde{g}'(r)|^{p-2} &= \frac{(p-1)^{p-2}}{p^{p-2} r^{p-2}} \left(\frac{N-1}{p-1} r \coth r - 1 \right)^{p-2} \tilde{g}^{p-2}(r) \\ &\geq \frac{(p-1)^{p-2}}{p^{p-2}} \frac{\tilde{g}^{p-2}(r)}{r^{p-2}}.\end{aligned}\tag{7.10}$$

Substituting (7.10) in (7.8) we conclude

$$\begin{aligned}-\Delta_p \tilde{g} - \left(\frac{N-1}{p} \right)^p H_p(r) \tilde{g}^{p-1} &\geq \frac{(p-1)^p}{p^p} \frac{1}{r^p} \tilde{g}^{p-1} \\ &+ \frac{(p-1)^{p-1} (p-2)(N-1)}{p^p} \left(\frac{\coth r}{r} \right) \frac{1}{r^{p-2}} \tilde{g}^{p-1} \\ &+ \frac{(N-1)(N-1-p(p-1))}{p^2} \frac{1}{\sinh^2 r} \tilde{g}^{p-1} \\ &\geq \frac{(p-1)^{p-1} (N(p-2)+1)}{p^p} \frac{1}{r^p} \tilde{g}^{p-1} \\ &+ \frac{(N-1)(N-1-p(p-1))}{p^2} \frac{1}{\sinh^2 r} \tilde{g}^{p-1}.\end{aligned}$$

This proves that $\tilde{g}(r) = \left(\frac{r}{\sinh r} \right)^{\frac{N-1}{p}} r^{\frac{p-N}{p}}$ is a super-solution of the equation corresponding to (7.1). Hence, by Allegretto-Piepenbrink Theorem inequality (2.5) follows immediately.

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